Derivatives in Finance: Understanding the Interest Rate Swap

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Introduction

In the world of modern finance, interest rates play a fundamental role in determining the cost of borrowing and the return on savings or investments. They influence the decisions of households, corporations, governments, and financial institutions. Managing exposure to interest rate fluctuations has therefore become a central objective in risk management and asset-liability strategies. Among the most widely used instruments to achieve this goal, the interest rate swap holds a prominent position.

An interest rate swap is, in essence, a contract between two parties who agree to exchange periodic interest payments, calculated on a notional principal, over a specified period of time. The key aspect of this contract lies in the nature of the interest rates being exchanged: typically, one party commits to paying a fixed rate, while the other pays a floating rate, which fluctuates over time according to a reference benchmark such as LIBOR or SOFR. The underlying notional principal serves only as a reference for the calculations; it is never exchanged between the parties.

This structure allows financial actors to tailor their exposure to interest rates without modifying the underlying instruments they hold. For this reason, interest rate swaps are widely used by corporations, banks, pension funds, and even governments to manage risk, enhance flexibility, and align financial strategies with market expectations.

Main Floating Rate Benchmarks: LIBOR and SOFR

To fully understand how interest rate swaps function in practice, it is essential to grasp the nature of the floating rate typically involved in such contracts. Historically, the most widely used benchmark for floating-rate payments has been the London Interbank Offered Rate (LIBOR). This rate was designed to reflect the average interest rate at which major global banks were willing to lend unsecured funds to one another over various short-term maturities and in different currencies. Published daily, LIBOR served for decades as the foundation of an immense array of financial instruments, including loans, bonds, mortgages, and derivatives.

However, LIBOR had a serious structural weakness: it was not based on real transactions but on estimates submitted by a panel of banks. This made it vulnerable to manipulation, as revealed by a series of scandals that undermined trust in the rate. As a result, regulators around the world initiated a transition toward more robust and transparent benchmarks.

In the United States, the primary successor to LIBOR is the Secured Overnight Financing Rate (SOFR). SOFR is based on actual overnight borrowing transactions in the U.S. Treasury repurchase agreement (repo) market, where institutions borrow funds using Treasury securities as collateral. This makes SOFR grounded in real market activity, with high volumes and minimal credit risk. Its secured nature also explains why SOFR tends to be lower than LIBOR, which incorporated a credit premium.

Unlike LIBOR, SOFR is not forward-looking and does not provide rates for multiple maturities (such as 1 month, 3 months, or 6 months). Instead, it is an overnight rate, and term structures must be constructed synthetically if needed. Nevertheless, its transparency and reliability have led to its adoption as the new standard benchmark for U.S. dollar-based floating rate instruments, including interest rate swaps.

A Simple Story: Why Use an Interest Rate Swap?

To grasp the intuition behind an interest rate swap, let us consider a simple and relatable situation. Imagine a company named Alpha Ltd. that has issued debt with a floating interest rate, for example, SOFR plus 2%. This means that every six months, the company must pay interest based on whatever the current SOFR rate is, plus a fixed margin of 2%. While this may be beneficial when interest rates are low, it introduces a significant risk: if interest rates rise, the company will face higher and potentially unpredictable interest expenses.

On the other side of the market, suppose there is an investor, perhaps a pension fund, who has invested in long-term government bonds that pay a fixed interest rate, say 5% per year. This investor may believe that interest rates are about to fall, making fixed-rate investments less attractive. However, their current portfolio is locked in at 5%, and altering it could involve selling assets with unfavorable market conditions or capital losses.

Now imagine that these two entities, Alpha Ltd. and the pension fund, enter into a swap agreement. Alpha agrees to pay the pension fund a fixed rate of 5% annually on a notional of €10 million. In return, the pension fund agrees to pay Alpha the floating rate, SOFR + 2%, on the same notional. The effect of this agreement is transformative: Alpha has effectively converted its floating-rate debt into fixed-rate debt, achieving predictability in its interest payments. The pension fund, conversely, has transformed part of its fixed-rate income into a floating-rate stream, giving it exposure to future changes in interest rates. This transaction does not require either party to change their existing debt or asset holdings. Instead, it overlays a synthetic structure that aligns their respective financial needs.

Cash Flows and Valuation

Let us now examine how the cash flows work in practice. Consider a swap with a notional of €1,000,000 and a duration of three years, with annual payments. Suppose the fixed rate agreed upon is 5%, and the floating leg is based on the prevailing SOFR rate plus a spread of 2%.

In the first year, assume the SOFR rate is exactly 3%. This implies that the floating payment due is 5%, which exactly matches the fixed rate. Consequently, there is no net payment between the parties.

In the second year, suppose that SOFR increases to 4%. The floating leg now amounts to 6%, while the fixed leg remains at 5%. The party paying floating owes more than the one paying fixed, so the net difference, 1% of the notional, is paid to the fixed payer. This results in a cash payment of €10,000.

In the third year, assume SOFR falls to 2.5%, leading to a floating rate of 4.5%. The fixed leg is now higher, and the fixed payer owes 0.5% of the notional to the floating payer, resulting in a €5,000 payment in the opposite direction.

These net settlements are the essence of the swap's operation. It is important to note that only the difference is paid; the two gross legs are not exchanged in full, reducing cash flow needs and simplifying accounting.

Mathematical Formulation and Fair Pricing

As introduced in Post 7 of this series, where we explored the pricing of forward contracts, a fundamental principle in modern financial theory is the **Fundamental Theorem of Finance**. According to this theorem, in an arbitrage-free market, the price of any financial instrument is given by the expected value of its future cash flows, discounted under the risk-neutral measure $\mathbb{Q}$. Mathematically, if an asset delivers a cash flow $X(T)$ at time T, and $G(t)$ is a strictly positive numéraire process, its price at time t is:

$$\text{Price}_t = \mathbb{E}_t^{\mathbb{Q}}\left[X(T) \cdot \frac{G(t)}{G(T)}\right].$$

This pricing principle applies not only to simple one-time payoffs like forward contracts but also to continuous cash flows such as those in an interest rate swap. In this section, we aim to compute the fair fixed rate of a plain vanilla interest rate swap in continuous time. The swap exchanges a fixed rate for a floating rate over a continuous time interval from $t = 0$ to T.

We assume that the floating leg pays the (possibly stochastic) short rate process $r(s)$, while the fixed leg pays a constant rate r_{swap}, known as the fixed rate of the swap. The notional is N, and for simplicity, we set $N = 1$ throughout this section. Our goal is to determine the fair value of r_{swap} such that the contract has zero value at inception.

Pricing the Two Legs

Under the risk-neutral measure $\mathbb{Q}$, the present value at time zero of the floating leg is given by:

$$\text{FloatingLeg}(0) = \mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T r(s) \cdot \frac{G(0)}{G(s)}\, ds \right].$$

This is the expected discounted value of the stochastic short rate integrated over time. The fixed leg pays a deterministic continuous flow r_{swap}, so its present value is:

$$\text{FixedLeg}(0) = r_{\text{swap}} \cdot \mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T \frac{G(0)}{G(s)}\, ds \right].$$

Fair Swap Rate

To ensure the swap has zero value at inception, we require that the two legs have equal present value. Thus, we set:

$$\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T r(s) \cdot \frac{G(0)}{G(s)}\, ds \right] = r_{\text{swap}} \cdot \mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T \frac{G(0)}{G(s)}\, ds \right].$$

Solving for r_{swap} gives the fair swap rate in continuous time:

$$r_{\text{swap}} = \frac{\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T r(s) \cdot \frac{G(0)}{G(s)}\, ds \right]}{\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T \frac{G(0)}{G(s)}\, ds \right]}.$$

This expression shows that the fair fixed rate of the swap is the discounted weighted average of the stochastic short rate $r(s)$, under the risk-neutral measure.

Special Case: Constant Short Rate

If the short rate is constant, i.e., $r(s) = r$ for all $s \in [0, T]$, then the numerator simplifies to:

$$\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^T r \cdot e^{-rs} ds \right] = r \cdot \int_0^T e^{-rs} ds = 1 - e^{-rT}.$$

And the denominator becomes:

$$\int_0^T e^{-rs} ds = \frac{1 - e^{-rT}}{r}.$$

Therefore, in the deterministic case:

$$r_{\text{swap}} = \frac{1 - e^{-rT}}{\frac{1 - e^{-rT}}{r}} = r.$$

This confirms that when the short rate is constant, the fair fixed rate of the swap is exactly equal to the risk-free rate.

A Practical Application: Managing Bank Risk

To demonstrate how interest rate swaps can be used to hedge risk in a realistic setting, we simulate the case of a commercial bank holding a portfolio of fixed-rate mortgage assets and funding itself through floating-rate liabilities. This scenario creates an interest rate mismatch: the bank earns a fixed interest income but is exposed to rising funding costs if short-term interest rates increase. Such asymmetry can compress the bank's net interest margin and threaten its financial stability.

To mitigate this risk, the bank enters into a **plain vanilla interest rate swap** in which it pays a fixed rate and receives a floating rate indexed to the short-term market rate. This synthetic overlay effectively transforms its fixed-rate asset into a floating-rate one, realigning its interest rate exposure with its liability structure.

In our simulation, the floating rate is modeled as a **Vasicek process** under the risk-neutral measure $\mathbb{Q}$, described by the stochastic differential equation:

$$dr(t) = a(b - r(t))\,dt + \sigma\,dW^{\mathbb{Q}}(t),$$

where:

- $a = 0.3$: the speed of mean reversion,
- $b = 0.04$: the long-term mean level (under $\mathbb{Q}$),
- $\sigma = 0.02$: the volatility of the short rate,
- $r(0) = 0.03$: the initial short rate.

We simulate 10000 paths of $r(t)$ over a five-year horizon with daily time steps (250 steps per year). The fair fixed rate r_{swap} is then computed using the continuous-time pricing formula derived earlier:

$$r_{\text{swap}} = \frac{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T r(s)e^{-\int_0^s r(u)\,du}ds\right]}{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\int_0^s r(u)\,du}ds\right]}.$$

This corresponds to the discounted expected value of the floating leg divided by the present value of the fixed leg.

```python
import numpy as np
import matplotlib.pyplot as plt
from google.colab import files

# Vasicek Parameters
a = 0.3
b = 0.04
sigma = 0.02
r0 = 0.03
T = 5.0
dt = 1/250
time_grid = np.linspace(0, T, int(T/dt) + 1)
```

```python

# Functions
def B(t): return (1 - np.exp(-a * t)) / a

def A(t):
    Bt = B(t)
    term1 = (Bt - t) * (a * b - (sigma ** 2) / (2 * a ** 2))
    term2 = (sigma ** 2) * Bt ** 2 / (4 * a)
    return term1 - term2

def P_0_t(t):  # Bond price at time 0 for maturity t
    return np.exp(-A(t) - B(t) * r0)

def E_r_t(t):  # Expected short rate under Q
    return b + (r0 - b) * np.exp(-a * t)

# Computing the integrals
discounts = P_0_t(time_grid)
expected_rates = E_r_t(time_grid)
numerator = np.trapz(discounts * expected_rates, time_grid)
denominator = np.trapz(discounts, time_grid)
r_swap = numerator / denominator
print(f"Swap rate (closed-form, Vasicek): {r_swap:.4%}")

# Plots
fig, axs = plt.subplots(3, 1, figsize=(12, 10))

# 1. Floating rate vs swap rate
axs[0].plot(time_grid, expected_rates, label=r'Floating Rate $r(t)$')
axs[0].axhline(y=r_swap, color='red', linestyle='--', linewidth=2,
               label=fr'Fixed Swap Rate $r_{{\text{{swap}}}}$ = {r_swap:.2%}')
axs[0].set_title('Simulated Short Rate and Fixed Swap Rate', fontsize=14)
axs[0].set_ylabel('Interest Rate', fontsize=12)
axs[0].legend()
axs[0].grid(True)

# 2. Net flow
net_diff = expected_rates - r_swap
axs[1].plot(time_grid, net_diff, label=r'$r(t) - r_{\text{swap}}$', color='orange')
axs[1].axhline(0, color='black', linestyle='--', linewidth=0.8)
axs[1].set_title('Instantaneous Net Cash Flow (Floating - Fixed)', fontsize=14)
axs[1].set_ylabel('Net Flow', fontsize=12)
```

```
axs[1].legend()
axs[1].grid(True)

# 3. Cumulative cash flow
cumulated_cf = np.cumsum(net_diff * dt)
axs[2].plot(time_grid, cumulated_cf, label='Cumulative Net Swap Flow', color='green')
axs[2].axhline(0, color='black', linestyle='--', linewidth=0.8)
axs[2].set_title('Cumulative Swap Cash Flow Over Time', fontsize=14)
axs[2].set_xlabel('Time (Years)', fontsize=12)
axs[2].set_ylabel('Cumulative Value', fontsize=12)
axs[2].legend()
axs[2].grid(True)

plt.tight_layout()

# Save PDF and download
pdf_filename = "swap_simulation_plot.pdf"
fig.savefig(pdf_filename)
files.download(pdf_filename)
```

Listing 1: Swap Rate and Vasicek Closed-Form Simulation

We then select a single simulated path of $r(t)$ and analyze how this floating rate compares to the fixed swap rate over time. Figure 1 is used to illustrate the outcome of the hedging strategy:

- **Top plot:** shows the evolution of the simulated short rate $r(t)$ over five years alongside the fixed swap rate r_{swap}, represented as a horizontal dashed line. When the floating rate is above the fixed rate, the bank receives a net benefit from the swap.
- **Middle plot:** displays the instantaneous net cash flow $r(t) - r_{\text{swap}}$, which represents the difference between what the bank receives and pays in the swap at each moment in time.
- **Bottom plot:** shows the cumulative net cash flows generated by the swap over time. A positive value indicates that the swap has provided a net financial gain to the bank, helping offset higher floating-rate funding costs.

This analysis highlights the practical effectiveness of interest rate swaps in reducing exposure to interest rate fluctuations. By simulating the short rate under the risk-neutral measure and computing the fair fixed rate accordingly, we obtain a fully consistent pricing and risk-management framework. The results reinforce the idea that swaps are not only valuation tools but also fundamental instruments in strategic financial planning.

Conclusion

Interest rate swaps stand as a cornerstone of modern financial engineering, allowing institutions and investors to manage interest rate risk without altering the underlying composition

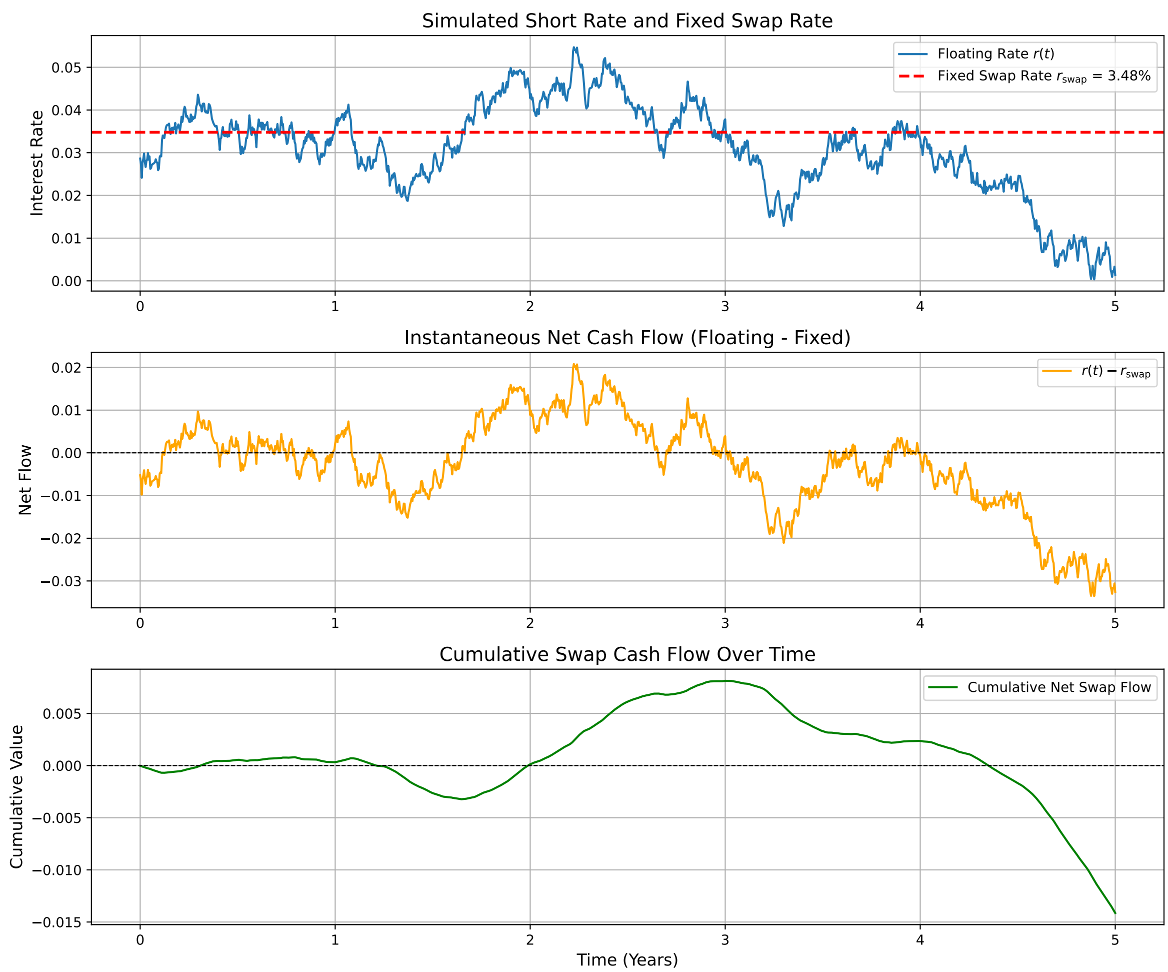

Figure 1: Simulated Vasicek short rate, fixed swap rate, net cash flow, and cumulative swap gains over time.

of their portfolios. Through both theoretical formulations and practical simulations, we have shown how these derivatives operate, how their fair value is determined under risk-neutral pricing, and how they can be used in realistic scenarios such as bank balance sheet management.

Using the Vasicek model for interest rates, we derived a closed-form representation of the fair swap rate and illustrated its implications via Python simulations. Our analysis demonstrated that interest rate swaps not only offer hedging capabilities but also serve as instruments of strategic alignment between assets and liabilities.

Understanding the mechanics and valuation of swaps is essential for financial practitioners, policymakers, and academics alike. As benchmark reforms transition markets from LIBOR to SOFR and other secured alternatives, mastering these tools remains more relevant than ever in ensuring robust and transparent financial systems.