From Payoffs to Pricing: A Deep Dive into Options and the Black-Scholes Formula

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1 Introduction

An option is a derivative contract that grants one of the counterparties the right, but not the obligation, to buy or sell an underlying asset at a predetermined price agreed upon at the inception of the contract. The key feature of this type of contract is that the party purchasing the option has the right, but not the obligation, to execute the transaction. In other words, they may choose to buy or sell the underlying asset, but only if they wish to do so (technically, if the option is *in the money*).

To understand the economic motivation behind the use of options, consider the following example. Suppose there are two market participants: **Company A**, an industrial firm that uses oil for its operations, and **Company B**, an oil supplier. Company A is concerned that oil prices may rise over the next months and would like to secure the right to purchase a given amount of oil at a future time T at a price fixed today, denoted by K.

Company B is willing to deliver the oil at time T, but wants to avoid the risk that committing today to a fixed price might be unfavorable if the market price rises significantly. Therefore, a forward contract may be too rigid for both parties.

A more flexible alternative is to use a **call option**. Company A can purchase a European call option that grants the right, but not the obligation, to buy oil at price K at maturity T. The price paid today for this flexibility is the *option premium*, which compensates Company B for the potential upside it gives away.

At maturity:

- If $S(T) > K$, where $S(T)$ is the market price at time T, Company A exercises the option and purchases oil below market value.
- If $S(T) \leq K$, Company A lets the option expire and buys oil on the market, incurring only the cost of the premium.

This arrangement offers price protection to the buyer while providing income to the seller. The asymmetry of rights and obligations is the essence of what makes options valuable.

Options can be broadly categorized into two main types:

- **Call Option**: Gives the buyer the right to purchase the underlying asset at a specified price, known as the strike price.
- **Put Option**: Gives the buyer the right to sell the underlying asset at the strike price.

Up to this point, we have intentionally remained generic with respect to the timing of the contract. However, different types of options specify different exercise conditions. For example:

- A **European option** can be exercised only at a specific maturity date T.
- An **American option** can be exercised at any time up to and including the maturity date.

Because an option grants a right without an obligation, its value is always strictly positive at inception. This distinguishes options from forward contracts, which, by construction, have zero value at inception. In contrast, purchasing an option requires paying a positive amount upfront, known as the *option premium*. The premium reflects the value of flexibility and asymmetry embedded in the contract, and it can be computed using principles from the **Fundamental Theorem of Finance**, which links the absence of arbitrage to pricing under a risk-neutral probability measure.

To determine the price of a financial asset, one must consider all the future cash flows it will, or may, generate. Let us focus on the case of a European call option, which grants the right to purchase an underlying stock S at a predetermined strike price K at maturity T. The value of the underlying at maturity is denoted by $S(T)$, and the option's payoff is given by $(S(T) - K)^+$, i.e., $S(T) - K$ if the option is exercised. This occurs only if the option is *in the money*, i.e., when $S(T) > K$.

We are now ready to write the pricing formula for a European call option under the risk-neutral measure $\mathbb{Q}$:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}}\left[(S(T) - K)\,\mathbb{I}_{\{S(T)>K\}}\,\frac{G(t)}{G(T)}\right],$$

where $\mathbb{I}_{\{S(T)>K\}}$ is the indicator function, equal to 1 when the option ends up in the money and 0 otherwise. The term $\frac{G(t)}{G(T)}$ denotes the discounting factor.

Similarly, the price of a European put option is given by:

$$P(t) = \mathbb{E}_t^{\mathbb{Q}}\left[(K - S(T))\,\mathbb{I}_{\{S(T)<K\}}\,\frac{G(t)}{G(T)}\right].$$

These pricing formulas form the foundation for deriving the well-known *put-call parity*. A key identity is:

$$\mathbb{I}_{\{S(T)>K\}} = 1 - \mathbb{I}_{\{S(T)<K\}}.$$

Substituting this into the call option formula, we obtain:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}} \left[(S(T) - K)(1 - \mathbb{I}_{\{S(T)<K\}}) \frac{G(t)}{G(T)} \right],$$

and expanding and rearranging terms:

$$C(t) = \underbrace{\mathbb{E}_t^{\mathbb{Q}} \left[(S(T) - K) \frac{G(t)}{G(T)} \right]}_{\textbf{(1)}} + \underbrace{\mathbb{E}_t^{\mathbb{Q}} \left[(K - S(T)) \, \mathbb{I}_{\{S(T)<K\}} \frac{G(t)}{G(T)} \right]}_{\textbf{(2)}}.$$

We immediately recognize that the second term corresponds to the price of the put option, $P(t)$. The first term can be further simplified. Assuming the underlying stock $S(t)$ does not pay dividends and that discounting is performed using a zero-coupon bond $B(t, T)$, we obtain:

$$\mathbb{E}_t^{\mathbb{Q}} \left[(S(T) - K) \frac{G(t)}{G(T)} \right] = \mathbb{E}_t^{\mathbb{Q}} \left[S(T) \frac{G(t)}{G(T)} \right] - \mathbb{E}_t^{\mathbb{Q}} \left[K \frac{G(t)}{G(T)} \right] = S(t) - KB(t, T),$$

where $B(t, T)$ denotes the price at time t of a zero-coupon bond maturing at time T, which pays 1 unit of currency at maturity.

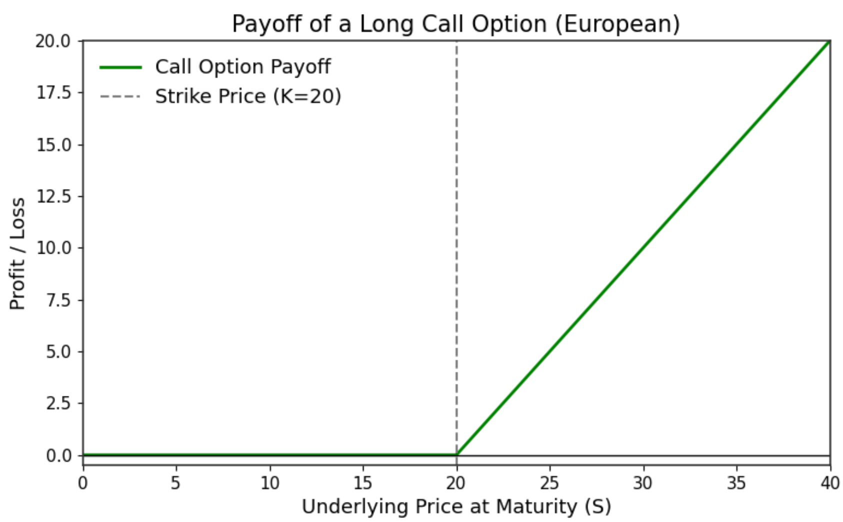

Figure 1: Payoff diagram of a **long call option**. The investor profits if the underlying asset's price at maturity exceeds the strike price.

Substituting back, we obtain the celebrated put-call parity:

$$C(t) = S(t) - KB(t, T) + P(t).$$

This relationship implies that any of the four instruments—call option $C(t)$, put option $P(t)$, underlying asset $S(t)$, and zero-coupon bond $B(t, T)$—can be replicated using the other three. This is a cornerstone result in derivatives pricing and reflects the internal consistency required by arbitrage-free markets.

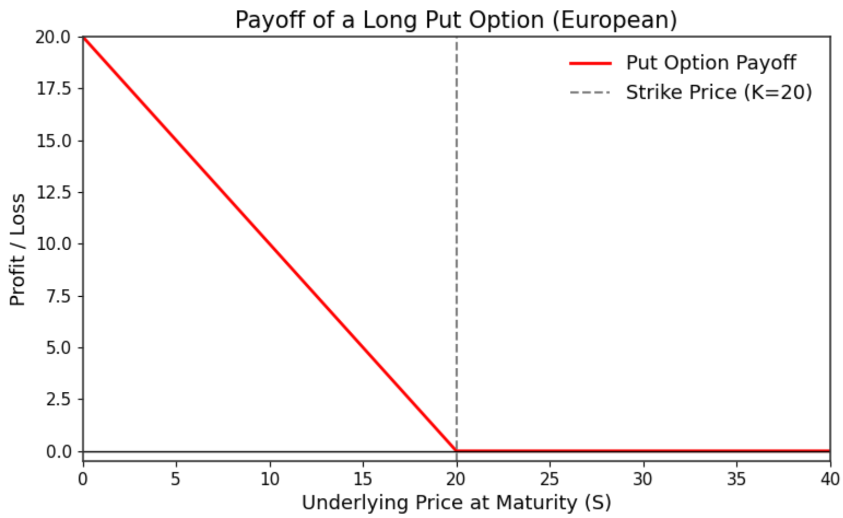

Figure 2: Payoff diagram of a **long put option**. The investor profits if the underlying asset's price falls below the strike price at maturity.

2 The Black-Scholes Model

How can we determine the fair price of an option under idealized market conditions?

One of the most celebrated answers is provided by the Black-Scholes model, developed by Fischer Black and Myron Scholes in 1973. The model offers a rigorous framework for pricing European-style options in continuous time, under the assumption that the price of the underlying asset follows a geometric Brownian motion:

$$dS(t) = S(t)\mu\, dt + S(t)\sigma\, dW(t),$$

where μ is the drift, σ is the volatility, and $W(t)$ is a standard Brownian motion under the historical probability measure $\mathbb{P}$. Simultaneously, the market includes a risk-free asset $G(t)$, whose return is constant over time:

$$dG(t) = G(t)r\, dt,$$

where r is the risk-free rate.

The probability that the underlying exceeds a strike K at maturity, under the historical measure, is:

$$\mathbb{P}\left(S(T) > K\right) = \mathbb{P}\left(S(t)e^{\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(W(T)-W(t))} > K\right).$$

Using algebra and the closed-form solution of $S(T)$, this can be rewritten as:

$$\mathbb{P}\left(-\frac{W(T)-W(t)}{\sqrt{T-t}} < \frac{\ln\left(\frac{S(t)}{K}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

and noting that the left-hand side is distributed as a standard normal, this probability can be evaluated using the standard normal cumulative distribution function.

Now consider the pricing of a European call option:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}} \left[(S(T) - K)\, \mathbb{I}_{\{S(T)>K\}} \cdot \frac{G(t)}{G(T)} \right],$$

and rearranging the terms:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}} \left[S(T)\, \mathbb{I}_{\{S(T)>K\}} \cdot \frac{G(t)}{G(T)} \right] - K\, \mathbb{E}_t^{\mathbb{Q}} \left[\mathbb{I}_{\{S(T)>K\}} \cdot \frac{G(t)}{G(T)} \right].$$

The expected value under a given probability measure (in this case $\mathbb{Q}$) of two generally dependent random variables can, in certain cases, be rewritten as the product of two expectations, where one is computed under a new probability measure. This is possible when one of the variables is used as a new numéraire, and the corresponding change of measure is defined via the Radon-Nikodym derivative.

In our case, we can write:

$$C(t) = \mathbb{S}\left(S(T) > K\right) \cdot S(t) - \mathbb{F}\left(S(T) > K\right) \cdot KB(t,T),$$

where the probabilities $\mathbb{S}$ and $\mathbb{F}$ are defined under the measures in which $S(t)$ and $B(t,T)$, respectively, serve as numeraires.

As a result, we can relate the Brownian motions under different measures using Girsanov's theorem:

$$dW^{\mathbb{Q}}(t) = \sigma(t)\, dt + dW^{\mathbb{S}}(t), \qquad dW^{\mathbb{Q}}(t) = \sigma_B(t,T)\, dt + dW^{\mathbb{F}}(t),$$

where $\sigma(t)$ is the volatility of the stock and $\sigma_B(t,T)$ is the volatility of the zero-coupon bond. In the Black-Scholes setting, $\sigma(t) = \sigma$ is constant and $\sigma_B(t,T) = 0$, since the risk-free rate is constant. This implies that $\mathbb{Q}$ and $\mathbb{F}$ coincide.

Now, using the stock dynamics under $\mathbb{P}$, and applying successive changes of measure, we have:

$$\begin{aligned} dS(t) &= S(t)\mu\, dt + S(t)\sigma\, dW(t) \\ &= S(t)r\, dt + S(t)\sigma\, dW^{\mathbb{Q}}(t) \\ &= S(t)(r + \sigma^2)\, dt + S(t)\sigma\, dW^{\mathbb{S}}(t), \end{aligned}$$

and alternatively:

$$dS(t) = S(t)r\, dt + S(t)\sigma\, dW^{\mathbb{F}}(t).$$

By substituting the appropriate drifts into the standard form of $S(T)$, we obtain:

$$\mathbb{S}\left(S(T) > K\right) = \mathcal{N}(d_1) = \mathcal{N}\left(\frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right),$$

and

$$\mathbb{F}\left(S(T) > K\right) = \mathcal{N}(d_2) = \mathcal{N}\left(\frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

which leads us to the famous Black-Scholes pricing formula:

$$C(t) = \mathcal{N}(d_1) \cdot S(t) - \mathcal{N}(d_2) \cdot KB(t,T).$$

To illustrate the use of this formula in practice, we implement the Black-Scholes pricing function in Python:

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm


# Black-Scholes formula
def bs_prices(S, K, T, r, sigma):
    """
    Parameters:
    - S: underlying asset price
    - K: strike price
    - T: time to maturity
    - r: risk-free rate
    - sigma: volatility
    Returns:
    - call: Call option price
    - put: Put option price (using the already shown put call parity)
    """
    d1 = (np.log(S / K) + (r + 0.5 * sigma**2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    call = S * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2)
    put = call + K * np.exp(-r * T) - S  # Put-call parity
    return call, put

# Parameters

K = 100        # Strike price
T = 2          # Time to maturity (years)
r = 0.01       # Risk-free rate
sigma = 0.2    # Volatility
S = 100        # Current asset price

# Compute prices given the previous example

call_price, put_price = bs_prices(S, K, T, r, sigma)
```

```python

print(f"Call Price: {call_price:.4f}")
print(f"Put Price:  {put_price:.4f}")
```

Listing 1: Black-Scholes Pricing in Python

The following parameters are used in the implementation:

- $S(t) = 100$
- $K = 100$
- $T = 2$ years
- $r = 0.01$
- $\sigma = 0.2$

The computed option prices are:

$$\text{Call price} = 12.1527, \qquad \text{Put price} = 10.1725.$$

The Black-Scholes model, while elegant and foundational, relies on several simplifying assumptions that limit its realism in practice. Chief among these is the assumption of a constant risk-free rate which does not reflect the dynamic nature of interest rates observed in real markets. Moreover, the model assumes that the underlying asset follows a geometric Brownian motion which implies normally distributed returns and constant volatility. Both assumptions are inconsistent with empirical evidence. In reality, asset returns exhibit fat tails, skewness and volatility clustering which the model cannot capture. These limitations have motivated the development of more advanced models that incorporate stochastic volatility, jumps and time-varying interest rates.

3 Popular Option Strategies

Beyond simple call and put positions, options can be combined in structured ways to create payoff profiles tailored to specific market views (very fancy ones exist). These combinations, known as *option strategies*, are widely used by traders and investors to manage risk, express directional views, or benefit from volatility, without necessarily having a strong opinion on the underlying asset's price level.

Some of the most well-known strategies include the straddle, strip and butterfly. These strategies involve buying and/or selling multiple options with different strike prices and sometimes different maturities. What makes them appealing is that they can shape the profit-and-loss distribution in highly customizable ways, allowing investors to profit from expectations about market movement, volatility, or lack thereof. We illustrate the payoff structures of three representative strategies in Figures 3, 4, and 5.

- *Straddle.* The straddle is a simple yet powerful strategy constructed by buying a call and a put option with the same strike price K and the same maturity T:

  $$C(K) + P(K).$$

  This strategy benefits from large movements in the underlying asset's price, regardless of whether they are upward or downward. It is particularly suitable for investors who anticipate high volatility but do not have a clear directional view. The payoff is symmetric around the strike and increases the further the price moves away from K, while the maximum loss is limited to the combined premiums paid.

- *Strip.* The strip is a variation of the straddle that places more weight on downward movements in the underlying asset. It consists of buying one call and two put options with the same strike price K and maturity T:

  $$C(K) + 2P(K).$$

  This setup increases exposure to negative price movements while still retaining upside potential. It is particularly suited to situations in which the investor anticipates high volatility, with a bias toward declining prices. The payoff profile remains convex but asymmetric, generating larger gains when the underlying falls significantly below the strike, and a more limited gain if the price increases.

- *Butterfly spread.* The butterfly is a strategy designed to profit when the price of the underlying remains within a relatively narrow range. It provides a limited but positive payoff if the underlying asset ends up between two strike prices, denoted K_1 and K_2. The strategy is most profitable when the price at maturity is exactly at the midpoint between the two. It is typically constructed using three call options:

  $$C(K_1) - 2C\left(\frac{K_1 + K_2}{2}\right) + C(K_2),$$

  where $C(K)$ denotes a call option with strike K. The result is a symmetric payoff profile with a peak centered at $\frac{K_1+K_2}{2}$, and a maximum loss equal to the net premium paid. The butterfly is well suited for low-volatility environments where the investor expects little deviation from a central value.

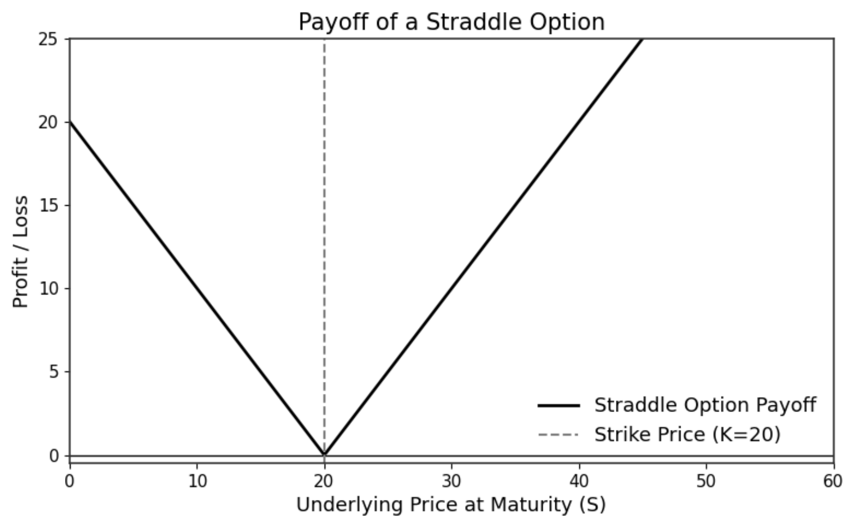

Figure 3: Payoff diagram of a straddle strategy. The position consists of buying a call and a put with the same strike price and maturity, and profits from large movements in either direction.

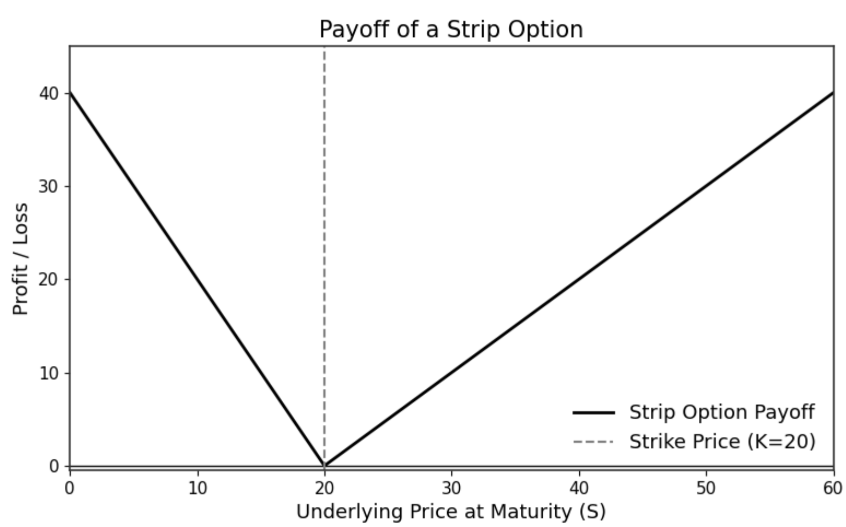

Figure 4: Payoff diagram of a strip strategy. The position includes one long call and two long puts with the same strike, resulting in a higher payoff when the underlying declines.

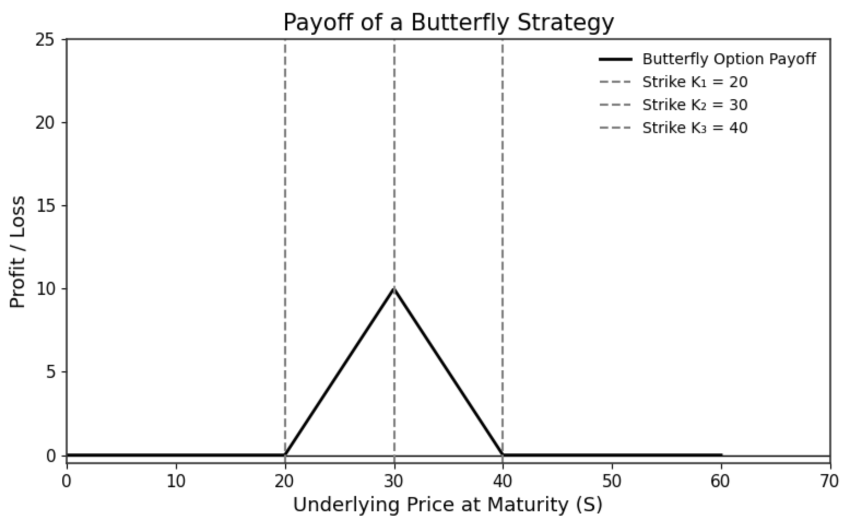

Figure 5: Payoff diagram of a butterfly spread. The strategy profits when the underlying ends near the middle strike and is commonly used in low-volatility environments.

4 Conclusion

In this work, we explored the fundamental role that options play in modern financial markets. Starting from the basic definition and historical context, we have shown how these derivatives empower market participants with flexible tools for hedging, speculation, and portfolio enhancement.

We presented the elegant pricing machinery introduced by the Black-Scholes model, which revolutionized the financial industry by providing a closed-form solution for the value of European call and put options. While based on idealized assumptions, the model remains a cornerstone of financial theory and a starting point for more advanced pricing frameworks.

Building on this foundation, we examined some of the most popular option strategies, demonstrating how traders and investors can craft tailored payoff profiles by combining different derivatives. These strategies, such as the straddle, strip, and butterfly spread, highlight the versatility of options in expressing complex market views, whether directional, volatility-based, or neutral.

Ultimately, options represent a powerful fusion of mathematical theory and financial intuition. Their proper use requires a deep understanding not only of pricing models, but also of market dynamics and investor objectives. As financial markets continue to evolve, so too will the tools and models we use to navigate them, and options will remain a key component of this ever-changing landscape.